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Lift-Curve Slope for Finite-Aspect-Ratio Wings

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Introduction

RECENTLY derived exact solutions for the lift produced by an elliptic planform flat plate in steady incompressible potential flow are applied to improve the commonly used approximations for the lift curve slope of finite aspect-ratio wings. An excellent approximation is derived for the initial lift-curve slope when the aspect ratio ≤ 2. For large aspect ratios, the lift-curve-slope equation is improved so that it can more accurately calculate the corresponding two-dimensional lift-curve slope from the wind tunnel test data of a finite-aspect-ratio wing.

Applications

The usual approximation for the lift curve slope C_{L_α} for incompressible flow on large aspect ratio wings is written as

$$C_{L_\alpha} = a_o/1 + \frac{57.3a_o}{\pi A e_1} \tag{1}$$

where the aspect ratio $A = b^2/S = (\text{span})^2/(\text{ref. area})$, $a_o \approx 0.1/$deg is the experimental lift curve slope for two-dimensional flow, and $e_1 \approx (1+\tau)^{-1}$ is either the experimental or theoretical correction for non-elliptical wing loadings. It will now be shown that a_o should not be included in the denominator so that a simpler and more accurate representation is given by

$$C_{L_\alpha} = \frac{a_o}{1 + (2/A)(1+\tau)} \tag{2}$$

This corresponds to the relations first given by Prandtl[1] with $\tau = 0$ for the first approximation to the elliptical spanwise lift distributions (also see Refs. 2 and 3).

The correct Eq. (2) is based on idealized lifting-line theory that has the wing downwash angle α_i remain constant along the entire wing span so one can write

$$C_L = 2\pi(\Delta\alpha - \alpha_i) \approx 2\pi[\Delta\alpha - (C_L/\pi A)(1+\tau)] \tag{3}$$

where $\Delta\alpha = (\alpha - \alpha_{L=0})$ is the angle of attack measured from zero lift, and τ is the correction to lifting-line theory for a non-elliptical spanwise lift distribution. Actually, α_i is constant only for the ideal elliptical load distribution when $\tau = 0$, so the τ correction is only an approximation based on 2π, which is not affected by a_o, a viscous real fluid effect on the two-dimensional flow. This approximation is clearly evident when one notes that if $\tau \neq 0$, then the induced drag coefficient becomes

$$C_{D_i} = (C_L^2/\pi A)(1+\delta) < C_L\alpha_i = (C_L^2/\pi A)(1+\tau) \tag{4}$$

For example, if $A = 6$ for a rectangular planform wing, then Glauert[3] calculated that $\delta = 0.046$ and $\tau = 0.163$ so that $C_{D_i} = 0.0555\ C_L^2 < C_L\alpha_i = 0.0617C_L^2$. This effect of $\tau \neq 0$ was first pointed out by Karman and Burgers.[4] The use of $a_o = 0.1$ in the denominator of Eq. (1) erroneously reduces the aspect ratio correction and increases C_{L_α}. For the same example of a rectangular wing with $A = 6$, the better approximation from Eq. (2) is $C_{L_\alpha} = 0.072$ for $a_o = 0.1$, whereas Eq. (1) gives a higher value of 0.074. The replacement of 2π by a_o is only justified in the numerator of Eq. (2) because it represents an additional correction for an increasing boundary layer thickness followed by flow separation, usually on the wing's upper surface, and is not related to $\alpha_i = (C_L/\pi A)(1+\tau)$.

It is important to note that the numerical values of τ are based on lifting-line theory, and there is an additional correction, even for the ideal elliptic planform wing, when one introduces the more exact lifting-surface theory. For example, Kida and Miyai[5] have shown that for the ideal potential flow about a thin flat plate with an elliptic planform, the second approximation for $A \gg 4/\pi$ can be written as

$$\tau = (8/\pi^2 A)[\ln(\pi A) - 1] + 0(1/A^2) \tag{5}$$

The best comparison for the restrictions upon Eq. (5) are given by Hauptman and Miloh[6] who derived the first explicit relations for the elliptic flat plate as given by

$$C_{L_\alpha} = 4\Big/\left[K + \frac{E^2(h)}{k + (\arcsin h)/h}\right];\quad \left(k = \frac{4}{\pi A} \leq 1\right) \tag{6}$$

$$C_{L_\alpha} = \left(\frac{32}{8+\pi^2}\right) = 1.791;\ \left(A = \frac{4}{\pi};\ k = 1;\ h = 0\right) \tag{7}$$

$$C_{L_\alpha} = \frac{4k}{1 + E^2(h)/[1 + (k^2/h)\ln(1/k + h/k)\ ]};\ (k = \pi A/4 \leq 1) \tag{8}$$

For Eq. (6), $k < 1$, $A = (4/\pi k)$ and $h^2 = (1 - k^2)$. For example, if $k = (\text{mid-chord/span}) = 1/2$ then $A = (8/\pi)$, $h = (\sqrt{3}/2)$ and $E(h) = E(60\ \text{deg}) = 1.211$, where E is the complete elliptic integral of the second kind. These equations are given here for completeness because Eq. (6) is incorrectly printed on page 771 (Eq. 34) of Ref. 7, and there is a misprint of Eq. (8) on pages 54 and 55 (Eq. 65a) of Ref. 6, where the term ahead of the $\ln$ term should be (k^2/h) and not (k/h).

The calculations from these equations are compared in Table 1, which also includes a very remarkable approximation derived by Helmbold,[8] which he gave as

$$C_{L_\alpha} = \frac{2\pi A}{2 + (4 + A^2)^{1/2}} \tag{9}$$

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Table 1 Comparison of C_{L_α} approximations for thin elliptic planform wings with $a_o = 2\pi$

A	Chord/span	h	$E(h)$	Eq. (2) $\tau = 0$	Eq. (2) $\tau =$ Eq. (5)	Eq. (9)	Exact solution Eqs. (6-8)
10	0.1273	0.9919	1.0240	5.236	5.068	5.151	5.063
$\frac{20}{\pi}$	$\frac{1}{5}$	$\sqrt{.96}$	1.0505	4.781	4.507	4.612	4.491
$\frac{8}{\pi}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1.2111	3.519	3.057	3.054	2.945
$\frac{4}{\pi}$	1	0	$\frac{\pi}{2}$	2.444	2.125	1.830	1.791
$\frac{2}{\pi}$	2	$\frac{\sqrt{3}}{2}$	1.2111	1.517	$\tau<0$	0.9759	0.9696

Helmbold[8] derived this equation by fixing the lifting-line vortex at the wing's quarter-chord and then evaluating the vortex strength by satisfying the surface boundary conditions only at the ¾-chord location, and finally taking the limit for $k = 4/(\pi A) \ll 1$. This approximation is remarkable because it reduces to Eq. (2) with $\tau = 0$ for large aspect ratios, and reduces to $C_{L_\alpha} = \pi A/2$ the exact limit for small A; and, as shown in Table 1, it provides a satisfactory approximation for the initial slope of the lift curve for all aspect ratios.

Hauptman[9] obtained the series expansion of Eq. (6) for $k = (4/\pi A) \ll 1$, up to the order of $(1/A^4)$ terms. His series development verified that Eqs. (2) and (5) do provide the correct expansion up to the order of $(1/A^2)$. Numerical calculations show that for $A \approx 6$, the values obtained from Eq. (5) are less than ½% greater than the exact values given by Eq. (6), and as shown in Table 1, Eq. (5) provides an excellent approximation for $A > (8/\pi)$. In addition, Hauptman's[9] series expansion can only be applied to $A > 2.235$ since his e/A^4 term has an increasingly large negative value for $\ln(\pi A) < 1.949$; this is not surprising because the expansion was based on $k = (4/\pi A) \ll 1$. However, Eq. (8) can provide a much more useful approximation for small aspect ratios, as indicated by Hauptman.[9] Unfortunately, his Eqs. (7) and (8) are both incorrect, and the desired expansion for $k = \pi A/4 \ll 1$ can be obtained by substituting the first line of Hauptman's Eq. (3) for $E(h)$, where $h = \sqrt{1-k^2}$, into our preceding Eq. (8) so as to obtain

$$C_{L_\alpha} = \pi A\left\{\left[1 + \frac{1}{2}\left(\ln\frac{4}{k} - \frac{1}{2}\right)k^2\right]^2\left[1 + k^2\ln\left(\frac{2}{k}\right) + \cdots\right]^{-1} + 1\right\}^{-1} = \pi A/[2 + (\ln 2 - 0.5)(\pi A/4)^2 + 0(k^4\ln^2 k)]$$

$$\approx \frac{\pi A/2}{1 + 0.05957\,A^2} \tag{10}$$

For $A \le 2$, Eq. (10) gives numerical values only slightly higher than the exact values obtained from either Eqs. (6) or (8). It is interesting to note that even though Eq. (10) was based on $k = (\pi A/4) \ll 1$ in Eq. (8), it gives an excellent numerical approximation for either Eqs. (6) or (8) as long as $A \le 2$. Equation (10) gives the initial lift curve slope $\pi A/2$ for a slender delta wing as the aspect ratio becomes small. However, there is a nonlinear α^2 effect that must be considered for all finite C_L.

Application to Large Aspect Ratio

For the usual case when $a_o < 2\pi$, Eq. (2) is more accurate than Eq. (1) for numerical calculations of the lift distributions on unswept large aspect ratio wings. Also, Eq. (2) simplifies the determination of a_o from wind tunnel tests with a finite aspect ratio since it gives more accurate values for a_0 as

$$a_o = C_{L_\alpha}[1 + (2/A)(1+\tau)] \tag{11}$$

On the other hand, Eq. (1) for the same data erroneously predicts that

$$a_o = C_{L_\alpha}[1 - C_{L_\alpha}(1+\tau)/(\pi A)]^{-1} \tag{12}$$

This gives too low a value for the two-dimensional lift curve slope (a_o) from the finite-aspect-ratio wind tunnel tests in the usual case when $a_o < 2\pi$.

Finally, the linearized Prandtl-Glauert subsonic compressibility correction for two-dimensional flow is directly applied to a finite aspect ratio wing by the use of Eq. (3) as

$$C_L = 2\pi(1-M^2)^{-½}[\Delta\alpha - (C_L/\pi A)(1+\tau)]$$

so that for large aspect ratios

$$C_{L_\alpha} = \frac{2\pi(1-M^2)^{-½}}{1 + (2/A)(1+\tau)(1-M^2)^{-½}}$$

As before, the correction for any real fluid viscous effects can now be approximated by replacing the theoretical 2π by $a_o \approx 0.1$/deg.

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